

Monotone trace functions of several variables

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Abstract

We investigate monotone operator functions of several variables under a trace or a trace-like functional. In particular, we prove the inequality $\tau(x_1 \cdots x_n) \leq \tau(y_1 \cdots y_n)$ for a trace τ on a C^* -algebra and abelian n -tuples $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ of positive elements. We formulate and prove Jensen's inequality for expectation values, and we study matrix functions of several variables which are convex or monotone with respect to the weak majorization for matrices.

1 Preliminaries

The question of monotonicity of operator functions under a trace or a trace-like functional was considered in [7]. In particular, conditions were given for which the implication

$$(1) \quad \underline{x} \leq \underline{y} \quad \Rightarrow \quad \varphi(f(\underline{x})) \leq \varphi(f(\underline{y}))$$

is valid for a positive functional φ on a C^* -algebra \mathcal{A} and abelian n -tuples \underline{x} and \underline{y} in \mathcal{A} contained in the domain of a function f of n variables.

Consider n -tuples $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$ of elements in a C^* -algebra \mathcal{A} and recall that \underline{x} is said to be abelian if the elements x_1, \dots, x_n are mutually commuting; and we write $\underline{x} \leq \underline{y}$ if $x_i \leq y_i$ for $i = 1, \dots, n$. We say that \underline{x} is in the domain of a real continuous function f of n variables defined on a cube $\underline{I} = I_1 \times \cdots \times I_n$ where each I_i is an interval and x_i is self-adjoint, if the spectrum $\sigma(x_i)$ of x_i is contained in I_i for $i = 1, \dots, n$. In this situation $f(\underline{x})$ is naturally defined as an element in \mathcal{A} .

*The author would like to dedicate this paper to the memory of Gert K. Pedersen.

It is proved in [7] that (1) holds, if f is continuous, convex and separately increasing, and the elements x_1, \dots, x_n are contained in the centralizer

$$\mathcal{A}^\varphi = \{y \in \mathcal{A} \mid \varphi(xy) = \varphi(yx) \forall x \in \mathcal{A}\}.$$

Likewise (1) holds, if f is continuous, concave and separately increasing, and the elements y_1, \dots, y_n are contained in the centralizer \mathcal{A}^φ . Finally (1) holds, if f is just continuous and separately increasing, the elements x_1, \dots, x_n and y_1, \dots, y_n are contained in the centralizer \mathcal{A}^φ , and \underline{x} and \underline{y} are compatible in the sense that the commutators satisfy

$$(2) \quad [x_i, y_j] = [x_j, y_i] \quad i, j = 1, \dots, n.$$

However, this last condition is very restrictive. It is equivalent to the demand that the line through \underline{x} and \underline{y} (or indeed just the midpoint) consists of abelian n -tuples.

It is the aim of the present article to remove the compatibility condition (2) and prove that (1) holds for a large class of separately increasing functions f and elements x_1, \dots, x_n and y_1, \dots, y_n in the centralizer \mathcal{A}^φ . In particular, we prove the implication

$$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \quad \Rightarrow \quad \tau(x_1 \cdots x_n) \leq \tau(y_1 \cdots y_n)$$

for a trace τ on a C^* -algebra \mathcal{A} and abelian n -tuples $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$ of positive elements in \mathcal{A} .

In section 3 we consider functionals which are given as the expectation value of a unit vector in a Hilbert space, and derive what we term Jensen's inequality for expectation values.

In section 4 we finally prove some related results for matrix functions of several variables, but with respect to the weak majorization for matrices.

2 Inequalities under a positive functional

Let \mathcal{C} be a separable abelian C^* -subalgebra of a C^* -algebra \mathcal{A} , and let φ be a positive functional on \mathcal{A} such that \mathcal{C} is contained in the centralizer \mathcal{A}^φ . The subalgebra is of the form $\mathcal{C} = C_0(S)$ for some locally compact metric space S , and by the Riesz representation theorem there is a finite Radon measure μ_φ on S such that

$$\varphi(y) = \int_S y(s) d\mu_\varphi(s) \quad y \in \mathcal{C} = C_0(S).$$

For each positive element x in the multiplier algebra $M(\mathcal{A})$ we have

$$0 \leq \varphi(xy) = \varphi(y^{1/2}xy^{1/2}) \leq \|x\|\varphi(y) \quad y \in \mathcal{C}_+.$$

The functional $y \rightarrow \varphi(xy)$ on \mathcal{C} consequently defines a Radon measure on S which is dominated by a multiple of μ_φ , and it is therefore given by a unique element $\Phi(x)$ in $L^\infty(S, \mu_\varphi)$. By linearization this defines a conditional expectation¹ of the multiplier algebra

$$(3) \quad \Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$$

such that

$$\int_S z(s)\Phi(x)(s) d\mu_\varphi(s) = \varphi(zx) \quad z \in \mathcal{C}, x \in M(\mathcal{A}).$$

In particular, $\Phi(z)(s) = z(s)$ almost everywhere in S for each $z \in \mathcal{C}$, cf. [10, 6, 7].

The following result is, although not explicitly stated, essentially proved in [7] and follows by inspection of the proof of Theorem 4.1 in the reference.

Theorem 2.1. *Let $f: \underline{I} \rightarrow \mathbf{R}$ be a continuous function defined on a cube $\underline{I} = I_1 \times \cdots \times I_n$ and let $\underline{x} = (x_1, \dots, x_n)$ be an abelian n -tuple in \mathcal{A} contained in the domain of f . If f is concave then*

$$\Phi(f(\underline{x})) \leq f(\Phi(x_1), \dots, \Phi(x_n))$$

almost everywhere, where $\Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$ is the conditional expectation in (3).

If in addition $y = (y_1, \dots, y_n)$ is an n -tuple in \mathcal{C} contained in the domain of f , and f is also separately increasing, then $\underline{x} \leq \underline{y}$ implies

$$(4) \quad \Phi(f(\underline{x})) \leq f(\Phi(x_1), \dots, \Phi(x_n)) \leq f(\Phi(y_1), \dots, \Phi(y_n)) = f(\underline{y})$$

almost everywhere, and consequently

$$\varphi(f(\underline{x})) = \varphi(\Phi(f(\underline{x}))) \leq \varphi(f(\underline{y})).$$

The monotonicity under φ is thus a trivial consequence of the assertion proved in equation (4), and it therefore comes as no surprise that we can find examples where (4) is violated, but we still have monotonicity under φ .

¹This is a slight abuse of language since the range is not a subalgebra of $M(\mathcal{A})$, but Φ is positive, linear and $\Phi(xy) = \Phi(x)y$ for $x \in M(\mathcal{A})$ and $y \in \mathcal{C}$.

Example 2.2. Let $\mathcal{A} = M_2$ be the C^* -algebra of 2×2 matrices, and let \mathcal{C} be the abelian subalgebra of the diagonal matrices. We consider the matrices

$$x = \begin{pmatrix} c & c \\ c & c \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} t & 0 \\ 0 & \lambda t \end{pmatrix} \quad 0 < c < t$$

and notice that $0 \leq x < y$ for $\lambda > c(t - c)^{-1}$. We choose the ordinary trace as the positive functional and obtain

$$\Phi(x^2) = \begin{pmatrix} 2c^2 & 0 \\ 0 & 2c^2 \end{pmatrix} \not\leq \begin{pmatrix} t^2 & 0 \\ 0 & \lambda^2 t^2 \end{pmatrix} = y^2 \quad \text{for } t < c\sqrt{2},$$

while as expected $\text{Tr } x^2 = 4c^2 < t^2(1 + c^2(t - c)^{-2}) \leq t^2(1 + \lambda^2) = \text{Tr } y^2$.

The theory of operator means [9] provides us with a key tool to obtain the implication in (1) for functions which are neither convex nor concave, provided we assume that all of the elements x_1, \dots, x_n and y_1, \dots, y_n are contained in the centralizer \mathcal{A}^φ .

Theorem 2.3. Let φ be a positive functional on a C^* -algebra \mathcal{A} , and let $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$ be abelian n -tuples of elements in the centralizer \mathcal{A}^φ . If $0 \leq x_i \leq y_i$ for $i = 1, \dots, n$ then

$$\varphi(x_1^{p_1} \cdots x_n^{p_n}) \leq \varphi(y_1^{p_1} \cdots y_n^{p_n})$$

for arbitrary non-negative exponents p_1, \dots, p_n .

Proof. The geometric mean $x \# y$ is defined for positive invertible elements $x, y \in \mathcal{A}$ by setting

$$x \# y = x^{1/2}(x^{-1/2}yx^{-1/2})^{1/2}x^{1/2}$$

and since

$$x^{1/2}(x^{-1/2}yx^{-1/2})^{1/2}x^{1/2} = \frac{1}{2\pi} \int_0^\infty 2(x^{-1} + \lambda y^{-1})^{-1} \lambda^{-1/2} d\lambda,$$

it follows that $x \# y$ is increasing in each variable. In fact, it can be extended to positive elements $x, y \geq 0$ in \mathcal{A} and becomes a concave and separately increasing function of the pair (x, y) , cf. [9]. Since x_1 and x_2 commute we therefore obtain

$$x_1^{1/2}x_2^{1/2} = x_1 \# x_2 \leq y_1 \# y_2 = y_1^{1/2}y_2^{1/2}.$$

We first note that $x_3^{1/2} \leq y_3^{1/2}$ by the Löwner-Heinz inequality. Furthermore, $x_3^{1/2}$ commutes with $x_1^{1/2}x_2^{1/2}$ (with the same statement for the y 's). We

may therefore apply the above procedure once more to the abelian pairs $(x_1^{1/2} x_2^{1/2}, x_3^{1/2})$ and $(y_1^{1/2} y_2^{1/2}, y_3^{1/2})$ to get

$$x_1^{1/4} x_2^{1/4} x_3^{1/4} \leq y_1^{1/4} y_2^{1/4} y_3^{1/4}$$

and by induction we finally obtain

$$x_1^{1/2^{n-1}} \cdots x_n^{1/2^{n-1}} \leq y_1^{1/2^{n-1}} \cdots y_n^{1/2^{n-1}}.$$

For any continuous and increasing function f defined on the positive half-axis we have

$$0 \leq x \leq y \quad \Rightarrow \quad \varphi(f(x)) \leq \varphi(f(y))$$

for elements x and y in the centralizer \mathcal{A}^φ , cf. [7, Theorem 4.2] (note that the compatibility condition in the reference is void for functions of one variable); cf. also [8, 5, 3]. Setting $f(t) = t^{2^{n-1} \cdot N}$ we therefore obtain

$$\varphi(x_1^N \cdots x_n^N) \leq \varphi(y_1^N \cdots y_n^N)$$

for arbitrary $N > 0$. Possibly by first applying Löwner-Heinz inequality for each entry we thus have

$$\varphi(x_1^{\alpha_1 N} \cdots x_n^{\alpha_n N}) \leq \varphi(y_1^{\alpha_1 N} \cdots y_n^{\alpha_n N})$$

for $\alpha_1, \dots, \alpha_n \in [0, 1]$ and $N > 0$, and since any set of positive exponents (p_1, \dots, p_n) can be written in this form the assertion follows. **QED**

3 Jensen's inequality for expectation values

Recall that a continuous field $t \rightarrow a_t$ of operators on a Hilbert space H defined on a locally compact Hausdorff space T equipped with a Radon measure ν is said to be a unital column field if

$$\int_T a_t^* a_t d\nu(t) = 1,$$

cf. [6].

Theorem 3.1. *Let $f : \underline{I} \rightarrow \mathbf{R}$ be a continuous convex function of n variables defined on a cube, and let $t \rightarrow a_t \in B(H)$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν . If $t \rightarrow \underline{x}_t$ is a bounded continuous field on T of abelian n -tuples of operators on H in the domain of f , then*

$$(5) \quad f((y_1 \xi \mid \xi), \dots, (y_n \xi \mid \xi)) \leq \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) \xi \mid \xi \right)$$

for any unit vector $\xi \in H$, where the n -tuple \underline{y} is defined by setting

$$\underline{y} = (y_1, \dots, y_n) = \int_T a_t^* \underline{x}_t a_t d\nu(t).$$

Proof. Set $\underline{x}_t = (x_{1t}, \dots, x_{nt})$ for $t \in T$ and consider the spectral resolutions

$$x_{it} = \int \lambda dE_{it}(\lambda) \quad i = 1, \dots, n; t \in T.$$

There is to each unit vector $\xi \in H$ a positive measure μ_ξ on \underline{I} such that

$$\mu_\xi(S_1 \times \dots \times S_n) = \int_T (E_{1t}(S_1) \cdots E_{nt}(S_n) a_t \xi \mid a_t \xi) d\nu(t)$$

for Borel sets $S_1 \subseteq I_1, \dots, S_n \subseteq I_n$ and since the column field $t \rightarrow a_t$ is unital, we obtain that μ_ξ is a probability measure. It satisfies

$$\int_T (g(\underline{x}_t) a_t \xi \mid a_t \xi) d\nu(t) = \int_{\underline{I}} g(\underline{s}) d\mu_\xi(\underline{s}) \quad \underline{s} = (s_1, \dots, s_n)$$

for any continuous function $g : \underline{I} \rightarrow \mathbf{R}$. In particular (putting $g_i(\underline{s}) = s_i$) we obtain

$$\int_T (x_{it} a_t \xi \mid a_t \xi) d\nu(t) = \int_{\underline{I}} s_i d\mu_\xi(\underline{s}) \quad i = 1, \dots, n.$$

We thus obtain

$$\begin{aligned} & f((y_1 \xi \mid \xi), \dots, (y_n \xi \mid \xi)) \\ &= f\left(\left(\int_T a_t^* x_{1t} a_t d\nu(t) \xi \mid \xi\right), \dots, \left(\int_T a_t^* x_{nt} a_t d\nu(t) \xi \mid \xi\right)\right) \\ &= f\left(\int_T (x_{1t} a_t \xi \mid a_t \xi) d\nu(t), \dots, \int_T (x_{nt} a_t \xi \mid a_t \xi) d\nu(t)\right) \\ &= f\left(\int_{\underline{I}} s_1 d\mu_\xi(\underline{s}), \dots, \int_{\underline{I}} s_n d\mu_\xi(\underline{s})\right) \\ &\leq \int_{\underline{I}} f(s_1, \dots, s_n) d\mu_\xi(\underline{s}) = \int_T (f(\underline{x}_t) a_t \xi \mid a_t \xi) d\nu(t) \\ &= \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) \xi \mid \xi\right), \end{aligned}$$

where we used spectral theory and the convexity of f .

QED

Remark 3.2. If we choose ν as a probability measure on T , then the trivial field $a_t = 1$ for $t \in T$ is unital and (5) takes the form

$$f\left(\left(\int_T x_{1t} d\nu(t)\xi \mid \xi\right), \dots, \left(\int_T x_{nt} d\nu(t)\xi \mid \xi\right)\right) \leq \left(\int_T f(\underline{x}_t) d\nu(t)\xi \mid \xi\right)$$

for fields of abelian n -tuples $\underline{x}_t = (x_{1t}, \dots, x_{nt})$ and unit vectors ξ . By choosing ν as an atomic measure with one atom we get a version

$$(6) \quad f((x_1\xi \mid \xi), \dots, (x_n\xi \mid \xi)) \leq (f(\underline{x})\xi \mid \xi)$$

of the Jensen inequality by Mond and Pečarić [12].

One may generalize Theorem 3.1 by using the notions and methods developed in the proof of [7, Theorem 3.1] to obtain the following result.

Theorem 3.3. Let \mathcal{C} be a separable abelian C^* -subalgebra of a C^* -algebra \mathcal{A} , let φ be a positive functional on \mathcal{A} such that \mathcal{C} is contained in the centralizer \mathcal{A}^φ and let

$$\Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$$

be the conditional expectation defined in (3). Let furthermore $f: \underline{I} \rightarrow \mathbf{R}$ be a continuous convex function of n variables defined on a cube, and let $t \rightarrow a_t \in M(\mathcal{A})$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν . If $t \rightarrow \underline{x}_t$ is a bounded, weak* measurable field on T of abelian n -tuples in \mathcal{A} in the domain of f , then

$$f(\Phi(y_1), \dots, \Phi(y_n)) \leq \Phi\left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t)\right)$$

almost everywhere, where the n -tuple \underline{y} in $M(\mathcal{A})$ is defined by setting

$$\underline{y} = (y_1, \dots, y_n) = \int_T a_t^* \underline{x}_t a_t d\nu(t).$$

Note that Theorem 3.1 follows from the preceding theorem by choosing φ as the trace and letting \mathcal{C} be the C^* -algebra generated by the orthogonal projection on the vector ξ .

4 Weak majorization for matrices

We consider a Hilbert space H of finite dimension m and introduce for any self-adjoint operator $x \in B(H)$ the m -tuple $(x_{[1]}, \dots, x_{[m]})$ of eigenvalues of x counted with multiplicity and ordered in a decreasing sequence. The notion of weak majorization for matrices was considered by Ando [1] and Bhatia [4].

Definition 4.1. Let x and y be self-adjoint operators on a Hilbert space H of finite dimension m . We say that x is weakly majorized by y , and we write $x \prec_w y$ if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$$

for $k = 1, \dots, m$.

The following result is known as Ky Fan's maximum principle, cf. Bhatia [4, p. 35].

Lemma 4.2. Let x be a self-adjoint operator on a Hilbert space H of finite dimension m , and take a natural number $k \leq m$. Then

$$\sum_{i=1}^k (xu_i \mid u_i) \leq \sum_{i=1}^k x_{[i]}$$

for any orthonormal set (u_1, \dots, u_k) of vectors in H .

Theorem 4.3. Let $f : \underline{I} \rightarrow \mathbf{R}$ be a convex function of n variables defined on a cube, let H be a Hilbert space of finite dimension and let $t \rightarrow a_t \in B(H)$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν . If $t \rightarrow \underline{x}_t$ is a bounded continuous field on T of abelian n -tuples of operators on H in the domain of f , then the inequality

$$f \left(\int_T a_t^* \underline{x}_t a_t d\nu(t) \right) \prec_w \int_T a_t^* f(\underline{x}_t) a_t d\nu(t)$$

is valid provided

$$\underline{y} = (y_1, \dots, y_n) = \int_T a_t^* \underline{x}_t a_t d\nu(t)$$

is an abelian n -tuple.

Note that the assumption of \underline{y} being an abelian n -tuple is void for functions of one variable.

Proof. For $m = \dim H$ we choose an orthonormal m -tuple (u_1, \dots, u_m) of common eigenvectors for the commuting matrices y_1, \dots, y_n in such a way that the corresponding eigenvalues of $f(\underline{y})$ are ordered in a decreasing se-

quence. We then obtain

$$\begin{aligned}
& \sum_{i=1}^k f \left(\int_T a_t^* \underline{x}_t a_t d\nu(t) \right)_{[i]} = \sum_{i=1}^k f(\underline{y})_{[i]} = \sum_{i=1}^k (f(\underline{y}) u_i \mid u_i) \\
&= \sum_{i=1}^k f((y_1 u_i \mid u_i), \dots, (y_n u_i \mid u_i)) \\
&\leq \sum_{i=1}^k \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) u_i \mid u_i \right) \\
&\leq \sum_{i=1}^k \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) \right)_{[i]} \quad k = 1, \dots, m,
\end{aligned}$$

where we used spectral theory, Jensen's inequality for expectation values (5) and Lemma 4.2. **QED**

Remark 4.4. *If we choose a probability measure ν and the trivial field $a_t = 1$ we obtain the inequality*

$$f \left(\int_T \underline{x}_t d\nu(t) \right) \prec_w \int_T f(\underline{x}_t) d\nu(t),$$

provided the integral $\int_T \underline{x}_t d\nu(t)$ is an abelian n -tuple.

The following result is a generalization to functions of several variables of a theorem by Aujla and Silva [2, Theorem 2.3] for functions of one variable.

Corollary 4.5. *Let $f : \underline{I} \rightarrow \mathbf{R}$ be a convex function of n variables defined on a cube \underline{I} , and let H be a Hilbert space of finite dimension. Then*

$$f(\lambda \underline{x} + (1 - \lambda) \underline{y}) \prec_w \lambda f(\underline{x}) + (1 - \lambda) f(\underline{y}) \quad \lambda \in [0, 1]$$

for compatible n -tuples \underline{x} and \underline{y} of operators on H in the domain of f .

Proof. Since \underline{x} and \underline{y} are compatible, the line through \underline{x} and \underline{y} consists of abelian n -tuples. The result therefore follows from the preceding remark by choosing a suitable atomic probability measure ν . **QED**

Theorem 4.6. *Let $f : \underline{I} \rightarrow \mathbf{R}$ be a convex and separately increasing function of n variables, and let H be a Hilbert space of finite dimension. Then*

$$\underline{x} \leq \underline{y} \quad \Rightarrow \quad f(\underline{x}) \prec_w f(\underline{y})$$

for abelian n -tuples $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$ of operators on H in the domain of f .

Proof. For $m = \dim H$ we choose an orthonormal m -tuple (u_1, \dots, u_m) of common eigenvectors for the commuting matrices x_1, \dots, x_n in such a way that the corresponding eigenvalues for $f(\underline{x})$ are ordered in a decreasing sequence. We then obtain

$$\begin{aligned} \sum_{i=1}^k f(\underline{x})_{[i]} &= \sum_{i=1}^k (f(\underline{x})u_i \mid u_i) = \sum_{i=1}^k f((x_1 u_i \mid u_i), \dots, (x_n u_i \mid u_i)) \\ &\leq \sum_{i=1}^k f((y_1 u_i \mid u_i), \dots, (y_n u_i \mid u_i)) \leq \sum_{i=1}^k (f(\underline{y})u_i \mid u_i) \\ &\leq \sum_{i=1}^k f(\underline{y})_{[i]} \quad k = 1, \dots, m, \end{aligned}$$

where we used spectral theory, the monotonicity and convexity of f , the inequality in (6) and Lemma 4.2 respectively. **QED**

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